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Generalization of Calabi-Yau/Landau-Ginzburg correspondence

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Abstract

We discuss a possible generalization of the Calabi-Yau/Landau-Ginzburg correspondence to a more general class of manifolds. Specifically we consider the Fermat type hypersurfaces M_N^k: $\sum_{i=1}^N X_i^k = 0$ in $\mathbf{CP}^{N-1}$ for various values of k and N. When $k < N$, the 1-loop beta function of the sigma model on M_N^k is negative and we expect the theory to have a mass gap. However, the quantum cohomology relation $\sigma^{N-1} = \text{const.}\sigma^{k-1}$ suggests that in addition to the massive vacua there exists a remaining massless sector in the theory if $k > 2$. We assume that this massless sector is described by a Landau-Ginzburg (LG) theory of central charge $c = 3N(1-2/k)$ with N chiral fields with $U(1)$ charge $1/k$. We compute the topological invariants (elliptic genera) using LG theory and massive vacua and compare them with the geometrical data. We find that the results agree if and only if k =even and N=even.

These are the cases when the hypersurfaces have a spin structure. Thus we find an evidence for the geometry/LG correspondence in the case of spin manifolds.

1 Introduction

It is well-known that it is possible to reproduce various geometrical data of Calabi-Yau manifolds by making using the orbifoldized Landau-Ginzburg theory. Such a correspondence between Calabi-Yau manifold and the Landau-Ginzburg theory has been known for quite some time [1, 2] and has been studied extensively in connection with mirror symmetry. A simple explanation of this correspondence was provided some time ago using the gauged linear sigma model with a Fayet-Illiopoulos parameter r [3]. In [3] it was shown that in the limit of large positive values of r the non-linear sigma model on the Calabi-Yau (CY) manifold is recovered while the theory reduces to a Landau-Ginzburg (LG) type model in the limit of large negative values of r. r has the meaning of the size of the Calabi-Yau manifold. Since the topological quantities remain invariant under a smooth variation of the Fayet-Illiopoulos parameter, one recovers topological invariants of CY manifolds by studying LG theories.

In this article we would like to generalize such a Calabi-Yau/Landau-Ginzburg correspondence to a more general class of manifolds. Specifically in the following we consider Fermat-type hypersurfaces M_N^k : $X_1^k + X_2^k + \cdots + X_N^k = 0$ in $\mathbf{CP}^{N-1}$ for various value of k and N. $k = N$ give the $N-2$-dimensional Calabi-Yau manifold. When $k < N$, the hypersurface has a positive 1st Chern class $N-k$ and is a typical Fano variety. We recall that the (one-loop) beta function of the supersymmetric non-linear sigma model on a manifold M is proportional to the minus of its 1st Chern class. We may then imagine that a quantum theory on Fano variety has a non-zero mass gap due to its asymptotic freedom.

However, the quantum cohomology relation (restricted to the Kähler subring) of the manifold M_N^k is known [4] and given by

$$\sigma^{N-1} = \beta k^k \sigma^{k-1} \tag{1.1}$$

where β is the world-sheet instanton amplitude (σ denotes the vacuum value of the scalar component of the vector multiplet in the linear σ model. It corresponds to the Kähler class of M_N^k). (1.1) generalizes the well-known relation for quantum cohomology ring of $\mathbf{CP}^{N-2}$

$$\sigma^{N-1} = \beta. \tag{1.2}$$

The above equation (1.1) strongly suggests that the theory actually consists of both massive and massless sectors if $k > 2$: (i) in the massive sector there exist $N-k$ split vacua corresponding to the roots of $\sigma^{N-k} = \beta k^k$ of (1.1). (ii) in addition the theory possesses a massless sector corresponding to the the degenerate solution $\sigma = 0$ of the relation (1.1). This massless sector should be described by some conformal field theory (CFT) in the infra-red limit.

We have to make an identification of this CFT describing the massless sector so that together with the massive sector it should reproduce the geometry of the hypersurface. A natural candidate for the CFT is given by the Landau-Ginzburg theory consisting of N chiral fields with the $U(1)$-charge $1/k$ corresponding to the defining equation $X_1^k + X_2^k + \cdots + X_N^k = 0$ of M_N^k. This choice seems natural since it gives back the standard CY-LG correspondence in the case $k = N$.

During the course of this work we became aware of an article by Witten [5] where he suggests an approach very similar to ours. He further discusses the case of $k > N$, hypersurfaces of general type, where the beta function is positive and the direction of renormalization group flow is inverted.

In the case of Fano varieties $N > k$, field theory on the manifold flows in the infra-red to LG theories. If one assigns the central charge of the free field theory on the manifold as $c_{UV} = 3(N-2)$, then the central charge decreases in the infra-red to $c_{IR} = 3N(1-2/k)$ of the LG-theory in accord with the theorem of Zamolodchikov. In the case of manifolds of general type $k > N$, one flows from the LG theory in the ultra-violet with $c_{UV} = 3N(1-2/k)$ to a free field theory in the infra-red with $c_{IR} = 3(N-2)$ and again the central charge decreases.

In the following by making use of the LG theory we compute various topological invariants for Fano and general hypersurfaces, i.e. Euler numbers, elliptic genera for the σ (signature) and $\hat{A}$-genus. Method of computation of the elliptic genera in CFT/LG theories are well-known and described in [6, 7, 8, 9, 10, 11]. We take N to be even so that the signature and $\hat{A}$-genus of the manifold are well-defined. We then compare the results with the geometrical computation of elliptic genera for the hypersurfaces. Geometrical method of computation may be found in the literature [12, 13].

It turns out that we find exact agreements between LG theory and geometry if and only if k=even. Note that with N=even the 1st Chern class $N-k$ is divisible by 2 when k is even and the manifold has a spin structure. Therefore we have found an evidence that the geometry/LG correspondence exists in a class of spin manifolds.

We have also examined a few examples of hypersurfaces in a weighted projective space and again verified the LG/geometry correspondence when the hypersurface has a spin structure.

2 Quantum cohomology relation for hypersurfaces

Let us start with the relation of quantum cohomology obtained by Collino and Jinzenji [4] for hypersurface. We first recall that the cohomology classes of hypersurfaces consist of two classes (1) and (2): elements in the class (1) are generated by the powers of the Kähler class ω and are given by $\{1, \omega, \omega^2, \cdots, \omega^{N-2}\}$. We call this class as the Kähler subring in the following. Elements in the class (2) are given by the (p,q)-forms at the middle dimension, $\{\mathcal{O}^{p,q},\ p+q = N-2\}$. Let us call them as primitive classes. Due to the Lefschetz theorem these elements exhaust the cohomology classes of the hypersurface. In the quantum cohomology at the genus=0 level these two classes are essentially decoupled from each other and one can restrict one's attention to the sector of Kähler subring.

It is then possible to derive the following relation for the degree k hypersurface ($k < N$)

$$\omega^{N-1} = \beta k^k \omega^{k-1}. \tag{2.3}$$

Here β denotes the basic 1-instanton amplitude. We note that eq.(2.3) generalizes the well-known formula for $\mathbf{CP}^{N-2}$

$$\omega^{N-1} = \beta. \tag{2.4}$$

When one uses the description of the supersymmetric linear sigma model [3], ω is replaced by the scalar field σ of the $\mathcal{N}=2$ vector multiplet in 2-dimensions.

Quantum cohomology relation for $\mathbf{CP}^{N-2}$ (2.4) may be interpreted as the equation for the minima of a perturbed superpotential $W = 1/N \cdot \sigma^N - \beta\sigma$. In this case the quantum deformation corresponds to the most relevant perturbation. At each of the $N-1$ minima $\{\sigma_j = \text{const} \cdot e^{2\pi i j/(N-1)},\ j = 0, 1, 2\cdots, N-2,\ \ \text{const} = \beta^{1/(N-1)}\}$, the second derivative of the superpotential does not vanish and the theory has a mass gap. Also in the case of $k=2$ the second derivative does not vanish at all minima and the system has a map gap.

However, in the case of larger values of $k \geq 3$ the perturbation becomes less relevant and the degeneracy of the vacua is not completely resolved. The perturbed superpotential $W = 1/N \cdot \sigma^N - \beta k^{k-1}\sigma^k$ possesses both the $N-k$ massive vacua at $\{\sigma_j = \text{const} \cdot e^{2\pi i j/(N-k)},\ j = 0, 1, 2\cdots, N-k-1,\ \ \text{const} = (\beta k^k)^{1/(N-k)}\}$ and the degenerate vacua at the origin $\sigma = 0$. Thus the system has both massive and massless degrees of freedom.

3 Landau-Ginzburg models

We assume that the massless degrees of freedom are described by the Landau-Ginzburg theory corresponding to the tensor product of N copies of $\mathcal{N}=2$ superconformal minimal models with level $k-2$. Central charge of this system is given by $c = 3N(1-2/k)$.

Let us first check if the sum of the number of massive vacua and the degenerate vacua of CFT add up to the Euler number of the hypersurface. We may use the $\mathcal{N}=2$ representation theory for the CFT sector, however, for the computation of topological invariants a simple free-field realization may be used.

Let us introduce the ratio of theta functions

$$Z_{LG}(\gamma) = \frac{1}{k}\sum_{a,b=0}^{k-1} n_{a,b} \left(\frac{\theta_1(\frac{(1-k)}{k}\gamma + \frac{b}{k} + \frac{a\tau}{k} \mid \tau\ )}{\theta_1(\frac{1}{k}\gamma + \frac{b}{k} + \frac{a\tau}{k} \mid \tau\ )} \right)^N \tag{3.5}$$

where θ_1 is defined as usual

$$\theta_1(z \,|\, \tau) = -iq^{1/8}(e^{\pi i z} - e^{-\pi i z}) \prod_{n=1} (1-q^n)(1-q^n e^{2\pi i z})(1-q^n e^{-2\pi i z}), \tag{3.6}$$
$$q = \exp(2\pi i \tau)$$

$n_{a,b}$ is a phase factor which will be determined by imposing modular invariance.

We also recall the definition of $\theta_i,\ i = 2, 3, 4$ functions

$$\theta_2(z \,|\, \tau) = q^{1/8}(e^{\pi i z} + e^{-\pi i z}) \prod_{n=1} (1-q^n)(1+q^n e^{2\pi i z})(1+q^n e^{-2\pi i z}), \tag{3.7}$$

$$\theta_3(z \,|\, \tau) = \prod_{n=1} (1-q^n)(1+q^{n-1/2} e^{2\pi i z})(1+q^{n-1/2} e^{-2\pi i z}), \tag{3.8}$$

$$\theta_4(z \,|\, \tau) = \prod_{n=1} (1-q^n)(1-q^{n-1/2} e^{2\pi i z})(1-q^{n-1/2} e^{-2\pi i z}). \tag{3.9}$$

Euler number

The above formula (3.5) corresponds to an amplitude

$$Z_{CFT} = \mathrm{Tr}(-1)^F e^{2\pi i\gamma J_0} q^{L_0} \bar{q}^{\bar{L}_0} \tag{3.10}$$

in the superconformal field theory where F is the sum of left and right-moving fermion numbers $F_L + F_R$ and J_0 is the zero mode of the $U(1)$ current in the left-handed sector. Z_{CFT} becomes the Euler number of the target manifold when $\gamma = 0$.

If we choose $\gamma = 0$ in the corresponding formula of LG theory (3.5), all the oscillator modes in fact cancel and we have contributions only from zeros modes. We set $n_{a,b} = 1$. Then we find

$$Z_{LG}(\gamma = 0) = \frac{(1-k)^N}{k} + \frac{k^2-1}{k}. \tag{3.11}$$

where the first term comes from the sector $a = b = 0$ (we use l'hopital's rule in evaluating this contribution) while each of the other sectors contributes $1/k$ to the second term.

The total number of ground states is the sum of eq.(3.11) and the number of massive vacua which is equal to $N - k$. Thus the LG prediction for the Euler number χ of M_N^k is given by

$$\chi = \frac{(1-k)^N + k^2 - 1}{k} + N - k = \frac{(1-k)^N + Nk - 1}{k}. \tag{3.12}$$

On the other hand, the Euler number of M_N^k is computed geometrically using the adjunction formula as

$$c_{N-2} = \left.\frac{(1+H)^N}{(1+kH)}\right|_{H^{N-2}} \times H^{N-2} = \frac{(1-k)^N - (1-kN)}{k^2} H^{N-2}. \tag{3.13}$$

Here H denotes the hyperplane class of M_N^k and c_{N-2} is the $(N-2)$-th Chern class. The symbol $|_{H^n}$ means to take the coefficient of the H^n term. Integrating c_{N-2} over the manifold and using $\int_{M_N^k} H^{N-2} = k$ we recover the LG prediction (3.12).

We may rewrite the above Euler number χ as

$$\chi = \frac{(1-k)^N + k - 1}{k} + N - 1 \tag{3.14}$$

In (3.14) we recognize the 2nd term as the number of elements in the Kähler subring and then the first term is identified as the number of primitive elements of M_N^k. On the other hand, in the LG description the first term is identified as the sum of contributions from the untwisted sectors $a = 0, b = 0, \cdots, k-1$ while the 2nd term equals the sum of contributions from the twisted sectors and the massive vacua. Thus we find the correspondence

$$\text{untwisted sector} \Longleftrightarrow \text{primitive classes,} \tag{3.15}$$

$$\text{twisted sector+massive vacua} \Longleftrightarrow \text{Kähler subring.} \tag{3.16}$$

We recognize that the quantum deformation takes place only in the sector of Kähler subring and the primitive classes remain intact under quantum deformation.

Hirzebruch signature

Let us now consider the Hirzebruch signature σ and its elliptic generalization. First we recall that the signature is well-defined only for $4n$ real dimensional manifolds. Thus we take N =even hereafter.

In the LG formulation the elliptic genus for the signature is given by (3.5) with $\gamma = 1/2$,

$$\chi_\sigma(\tau) = \frac{1}{k}\sum_{a,b=0}^{k-1} n_{a,b} \left(\frac{\theta_1(\frac{(1-k)}{2k} + \frac{b}{k} + \frac{a\tau}{k} \mid \tau)}{\theta_1(\frac{1}{2k} + \frac{b}{k} + \frac{a\tau}{k} \mid \tau)} \right)^N , \tag{3.17}$$

$$= \frac{(-1)^N}{k}\sum_{a,b=0}^{k-1} n_{a,b} \left(\frac{\theta_2(\frac{1}{2k} + \frac{b}{k} + \frac{a\tau}{k} \mid \tau)}{\theta_1(\frac{1}{2k} + \frac{b}{k} + \frac{a\tau}{k} \mid \tau)} \right)^N . \tag{3.18}$$

We assume that there is no contribution to the signature from the massive vacua.

First we study the modular property of (3.17) and determine the possible choice of phase factors $n_{a,b}$. It is known that the elliptic genus is a modular form under the group $\Gamma_0(2)$ which leaves invariant a fixed spin structure. $\Gamma_0(2)$ is the subgroup of $SL(2,\mathbf{Z})$ of index 6 and consists of matrices of the form

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ mod } 2 \right\}. \tag{3.19}$$

It is generated by the elements T^2, ST^2S^{-1} where S and T are the standard generators of $SL(2,\mathbf{Z})$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.20}$$

We recall the modular properties of $\theta_1(z|\tau)$,

$$\theta_1(z \pm \tau \mid \tau) = -exp(-i\pi(\tau \pm 2z))\theta_1(z \mid \tau), \tag{3.21}$$

$$\theta_1\left(\frac{z}{c\tau+d} \mid \frac{a\tau+b}{c\tau+d} \right) = (-i)^{1/2}(c\tau+d)^{1/2} \exp\left(\frac{i\pi c z^2}{c\tau+d}\right)\theta_1(z \mid \tau). \tag{3.22}$$

It is easy to see that under $T^2 : \tau \to \tau+2$, (3.17) transforms as

$$\chi_\sigma(\tau+2) = \frac{1}{k}\sum_{a,b=0}^{k-1} n_{a,b-2a} \left(\frac{\theta_1(\frac{(1-k)}{2k} + \frac{b}{k} + \frac{a\tau}{k} \mid \tau)}{\theta_1(\frac{1}{2k} + \frac{b}{k} + \frac{a\tau}{k} \mid \tau)} \right)^N . \tag{3.23}$$

On the other hand, under $ST^2S^{-1} : \tau \to \tau/(-2\tau+1)$, $\chi_\sigma(\tau)$ transforms as

$$\chi_\sigma(\frac{\tau}{-2\tau+1}) = \frac{1}{k}\sum_{a,b=0}^{k-1} n_{a,b} \left(\frac{\theta_1(\frac{(1-k)}{2k} + \frac{b}{k} + \frac{a}{k}\frac{\tau}{-2\tau+1} \mid \frac{\tau}{-2\tau+1})}{\theta_1(\frac{1}{2k} + \frac{b}{k} + \frac{a}{k}\frac{\tau}{-2\tau+1} \mid \frac{\tau}{-2\tau+1})} \right)^N \tag{3.24}$$

$$= \frac{1}{k}\sum_{a,b=0}^{k-1} n_{a,b}\exp\left[i\pi N(\frac{2a'}{k}+\frac{b}{k}+\frac{2-k}{2k}+\tau)\right] \tag{3.25}$$

$$\times\left(\frac{\theta_1(\frac{(1-k)}{2k}+\frac{b}{k}+\frac{a'}{k}\tau+\tau\mid\tau)}{\theta_1(\frac{1}{2k}+\frac{b}{k}+\frac{a'}{k}\tau\mid\tau)}\right)^N,\ a'=a-2b-1 \tag{3.26}$$

$$= \frac{i^N}{k}\sum_{a,b=0}^{k-1} n_{a+2b+1,b}\left(\frac{\theta_1(\frac{(1-k)}{2k}+\frac{b}{k}+\frac{a\tau}{k}\mid\tau)}{\theta_1(\frac{1}{2k}+\frac{b}{k}+\frac{a\tau}{k}\mid\tau)}\right)^N. \tag{3.27}$$

We find that there are two natural choices for the phase factor $n_{a,b}$

$$\begin{array}{ll} (i) & n_{a,b}=1, \\ (ii) & n_{a,b}=(-1)^a. \end{array} \tag{3.28}$$

Correspondingly χ_σ transforms under T^2 and ST^2S^{-1} as

$$\begin{array}{llll} (i) & \chi_\sigma & \longrightarrow & \chi_\sigma,\ \ (-1)^{N/2}\chi_\sigma \\ (ii) & \chi_\sigma & \longrightarrow & \chi_\sigma,\ \ -(-1)^{N/2}\chi_\sigma. \end{array} \tag{3.29}$$

If we examine the modular property of known formula for the elliptic genus, for instance, of K_3 surface [6], we find that the choice (ii) gives the correct transformation law. Thus we adopt $n_{a,b}=(-1)^a$ hereafter in this section,

$$\chi_\sigma(\tau) = -\frac{(-1)^{\frac{N}{2}}}{k}\sum_{a,b=0}^{k-1}(-1)^a\left(\frac{\theta_2(\frac{1}{2k}+\frac{b}{k}+\frac{a\tau}{k}\mid\tau)}{\theta_1(\frac{1}{2k}+\frac{b}{k}+\frac{a\tau}{k}\mid\tau)}\right)^N. \tag{3.30}$$

Here we also have adjusted the overall sign so that (3.30) agrees with the standard convention for the signature.

Let us first test (3.30) by comparing its ground state multiplicity with the classical value of the signature. In the case of $N=4$ (complex surface), for instance, it is easy to compute geometrically the signature of M_4^k if we use (4.53),(4.55) of the next section,

$$\sigma(M_4^k) = \frac{1}{3}(4-k^2)k. \tag{3.31}$$

$\sigma(M_4^4) = -16$ is the well-known value for the K_3 surface.

On the other hand, the degeneracy of the lowest state in LG theory (3.30) is given by

$$\sigma_{LG}(M_N^k) = -\frac{1}{k}\left(\sum_{b=0}^{k-1}\left(\frac{1+e^{-2\pi i(1/2k+b/k)}}{1-e^{-2\pi i(1/2k+b/k)}}\right)^N + k\sum_{a=1}^{k-1}(-1)^a\right) \tag{3.32}$$

where the first sum comes from the untwisted sector and the second from the twisted sectors. Note that the contributions from twisted sectors cancel when k =odd while they add up to $+1$ when k =even. We may rewrite the first sum as

$$-\sum_{n=1}^{N-1}(-1)^n\times\left(t^{\frac{1}{k}}\left(1+t^{\frac{1}{k}}+t^{\frac{2}{k}}+\cdots+t^{\frac{k-2}{k}}\right)\right)^N\Big|_{t^n}\,. \tag{3.33}$$

(3.33) is an alternating sum of coefficients of integer powers of t. In the case of $N=4$, for instance, after a simple calculation we find that (3.33) is equal to

$$\frac{1}{3}(4-k^2)k-1. \tag{3.34}$$

Thus

$$\sigma_{LG}=\sigma,\qquad k=\text{even} \tag{3.35}$$

while $\sigma_{LG}=\sigma-1,\ k=\text{odd}$. Therefore the prediction of the LG theory agrees with the geometry only when k =even. When k =odd, there is a missing factor of 1 in σ_{LG}.

More precisely we may identify the above coefficients in (3.33) as the the Hodge numbers. In the case of $N=4$

$$c_n=\left(t^{\frac{1}{k}}\left(1+t^{\frac{1}{k}}+t^{\frac{2}{k}}+\cdots+t^{\frac{k-2}{k}}\right)\right)^4\Big|_{t^n}=h^{3-n,n-1}-\delta_{n,2},\ \ n=1,2,3. \tag{3.36}$$

Here a factor 1 is subtracted from $h^{1,1}$ in order to eliminate the contribution from the Kähler class. In the case k =even this factor 1 together with an additional factor 1 from the twisted sectors adds up to 2 and one obtains

$$\sigma_{LG}=2+2h^{2,0}-h^{1,1} \tag{3.37}$$

in agreement with the index theorem. In the case k =odd, however, twisted sectors cancel and LG theory fails to reproduce (3.37). One can check that this phenomenon occurs for all values of N.

We may further test the agreement of the LG theory with the geometry at excited levels, in particular in the case of k =even. Unfortunately, it is not easy to evaluate the expression (3.30) exactly except for the trivial case $k=1$ where $\chi_\sigma(\tau)$ vanishes identically,

$$k=1:\quad \chi_\sigma(\tau)=0,\quad \text{any } N. \tag{3.38}$$

In the case k =even, however, we may repeatedly use the addition formula of the $\wp$ function and evaluate (3.30) in a closed form. Details are described in the appendix. Computation in the case $k=2$ is straightforward and is given by

$$k=2:\quad \chi_\sigma(\tau)=\begin{cases}0 & \frac{N}{2}\ \text{is even}\\ \\ 2 & \frac{N}{2}\ \text{is odd}\end{cases} \tag{3.39}$$

This is in accord with the fact that when $k = 2$, central charge of the CFT vanishes ($c = N(1 - 2/k) = 0$) and the theory does not possess excited states.

After somewhat lengthy computations we obtain expressions for $\chi_\sigma(\tau)$ for $k = 4$ and $k = 6$ with various values of N

$$k=4: \quad \chi_\sigma(\tau) = \begin{cases} -8\left(\dfrac{(\theta_3(0|\tau))^2}{(\theta_4(0|\tau))^2} + \dfrac{(\theta_4(0|\tau))^2}{(\theta_3(0|\tau))^2}\right) & N=4 \\[2ex] 100, & N=6 \\[2ex] -288\left(\dfrac{(\theta_3(0|\tau))^2}{(\theta_4(0|\tau))^2} + \dfrac{(\theta_4(0|\tau))^2}{(\theta_3(0|\tau))^2}\right), & N=8 \\[2ex] 4\left(681 + 80\dfrac{(\theta_3(0|\tau))^4}{(\theta_4(0|\tau))^4} + 80\dfrac{(\theta_4(0|\tau))^4}{(\theta_3(0|\tau))^4}\right), & N=10 \\[2ex] -8\left(\dfrac{(\theta_3(0|\tau))^2}{(\theta_4(0|\tau))^2} + \dfrac{(\theta_4(0|\tau))^2}{(\theta_3(0|\tau))^2}\right)\left(1193 + 16\dfrac{(\theta_3(0|\tau))^4}{(\theta_4(0|\tau))^4} + 16\dfrac{(\theta_4(0|\tau))^4}{(\theta_3(0|\tau))^4}\right), & N=12 \end{cases} \tag{3.40}$$

$$k=6: \quad \chi_\sigma(\tau) = \begin{cases} -32\left(\dfrac{(\theta_3(0|\tau))^2}{(\theta_4(0|\tau))^2} + \dfrac{(\theta_4(0|\tau))^2}{(\theta_3(0|\tau))^2}\right), & N=4 \\[2ex] 2\left(419 + 16\dfrac{(\theta_3(0|\tau))^4}{(\theta_4(0|\tau))^4} + 16\dfrac{(\theta_4(0|\tau))^4}{(\theta_3(0|\tau))^4}\right), & N=6 \\[2ex] -6272\left(\dfrac{(\theta_3(0|\tau))^2}{(\theta_4(0|\tau))^2} + \dfrac{(\theta_4(0|\tau))^2}{(\theta_3(0|\tau))^2}\right), & N=8 \end{cases} . \tag{3.41}$$

We find that the formulas for $k = 2, 4, 6$ all agree with the geometrical data presented in the next section.

On the other hand in the case of odd k we can not evaluate (3.30) in a closed form but only obtain its q-expansion. We list the formulas for $k = 3$ and 5

$$k=3: \quad \chi_\sigma(\tau) = \begin{cases} -6 - 48q - 240q^2 - 912q^3 + \cdots, & N=4 \\ 18 + 216q + 1512q^2 + 7848q^3 + \cdots, & N=6 \\ -54 - 864q - 7776q^2 - 50976q^3 + \cdots, & N=8 \end{cases} \tag{3.42}$$

$$k=5: \ \chi_\sigma(\tau) = \begin{cases} -36 - 464q - 3472q^2 - 19392q^3 + \cdots, & N=4 \\ 340 + 6600q + 70680q^2 + 548640q^3 + \cdots, & N=6 \\ -3220 - 83360q - 1162400q^2 + \cdots, & N=8. \end{cases} \tag{3.43}$$

By comparing these results with the geometrical data we conclude that the LG predictions reproduce the elliptic genus for signature exactly if and only if k = even. We note that the hypersurface M_N^k with N =even is a spin manifold when k =even and hence we find the agreement of LG theory with geometry when the target space is a spin manifold.

Condition of spin manifold seems natural since the signature is closely related to the $\hat{A}$ genus which exists only for the spin manifold. In the case of Calabi-Yau manifolds the elliptic genera for σ and $\hat{A}$ are in fact given by the same modular function evaluated at different values of its argument.

We note that in the previous section Euler number of the manifold was correctly computed using LG theory without any condition for N and k. In section 5 we find that the elliptic genus for $\hat{A}$ (which exists only for N =even and k =even) can also be correctly reproduced by LG theory. Thus LG theory works at

$$\begin{cases} \chi \text{ (exists for any } N \text{ and } k) & \text{LG works for any } N,\ k \\ \sigma \text{ (exists for } N = \text{even}) & \text{LG works for } N = \text{even and } k = \text{even} \\ \hat{A} \text{ (exists for } N = \text{even and } k = \text{even}) & \text{LG works for } N = \text{even and } k = \text{even.} \end{cases} \tag{3.44}$$

4 geometrical computation

Let us now turn to the geometrical evaluation of the elliptic genus in order to provide data to test LG predictions. Outline of calculation goes as follows.

Elliptic genus for the loop-space signature operator on a manifold M of complex dimension d is defined as [14, 15]

$$\sigma(\mathcal{L}M) = 2^d \int_M ch(\mathbf{E}_q) L(M) \tag{4.45}$$

where $ch(\mathbf{E}_q)$ is the Chern character for a bundle $\mathbf{E}_q$

$$\mathbf{E}_q = (\otimes_{n\geq 1}(\Lambda_{q^n} TM)) \otimes (\otimes_{n\geq 1}(\Lambda_{q^n} T^*M)) \otimes (\otimes_{n\geq 1} S_{q^n}(TM \oplus T^*M)). \tag{4.46}$$

Here $\Lambda_{q^n}(TM)$ means, for instance, $1 + q^n TM + q^{2n}\Lambda(TM)^2 + \cdots$ and $S(\Lambda)$ stands for the (anti-)symmetrization of tensor product of tangent bundles TM. T^*M is the dual of TM.

$L(M)$ is the Hirzebruch L-polynomial

$$L(M) = \prod_{i=1}^d \frac{(\frac{u_i}{2})}{\tanh(\frac{u_i}{2})} \tag{4.47}$$

where the tangent bundle TM is split into a sum of line bundles with 1st Chern classes $\{u_i, i = 1, \cdots, d\}$. Hirzebruch polynomial may be expanded as

$$\frac{\frac{u}{2}}{\tanh\frac{u}{2}} = \exp\Big(\sum_{j=1}^{\infty} (-1)^{j-1} \frac{(2^{2j-1}-1)}{j\cdot(2j)!} B_j u^{2j}\Big) \tag{4.48}$$

where B_j's are the Bernoulli numbers.

Next by introducing an auxiliary function

$$f(q,z)=\prod_{n=1}\frac{(1+e^{2\pi iz}q^n)(1+e^{-2\pi iz}q^n)}{(1-e^{2\pi iz}q^n)(1-e^{-2\pi iz}q^n)} \tag{4.49}$$

we define

$$F_{2j}(q)=\frac{1}{2}(\frac{1}{2\pi i}\frac{d}{dz})^{2j}\log f(q,z)\,|_{z=0} \quad j=1,2,3\cdots \tag{4.50}$$

We then have an identity

$$\exp(\frac{u}{2\pi i}\frac{d}{dz})f(q,z)|_{z=0}=f(q)\exp\left(\sum_{j=1}^{\infty}\frac{2}{(2j)!}F_{2j}(q)u^{2j}\right) \tag{4.51}$$

where

$$f(q)=f(q,z)|_{z=0}. \tag{4.52}$$

Then the elliptic genus is expressed as

$$\sigma(\mathcal{L}M)=\int_M(2f(q))^d\exp\left(\sum_{j=1}^{\infty}\frac{2}{(2j)!}\left((-1)^{j-1}(2^{2j-1}-1)\frac{B_j}{2j}+F_{2j}(q)\right)\left(\sum_{i=1}^{d}u_i^{2j}\right)\right). \tag{4.53}$$

In the case of the degree-k hypersurface in $\mathbf{CP}^{N-1}$ Potryagin classes are given by

$$\sum_{n=0}^{\infty}(-1)^np_n=\prod_{i=1}^{N-2}(1-u_i^2)=\frac{(1-H^2)^N}{(1-k^2H^2)} \tag{4.54}$$

where H is the hyperplane class. Thus by taking the logarithm of (4.54) we find

$$\sum_{i=1}^{N-2}u_i^{2j}=(N-k^{2j})H^{2j}. \tag{4.55}$$

It is then easy to generate q-series for $\sigma(\mathcal{L}M)$ for various values of k and N. It turns out that again in the case k =even it is possible to obtain a closed formula for the elliptic genus. We have evaluated them in the case of $k=2,4,6$ and reproduced exactly the results of the LG theory (3.39),(3.40),(3.41). Details are given in the appendix.

On the other hand in the case k =odd we obtain only the q-series. We present the results of $k=1,3,5$,

$$k=1: \quad \sigma(\mathcal{L}M)=\begin{cases}1+32q+256q^2+1408q^3+\cdots, & N=4\\ 1+96q+2304q^2+28800q^3+\cdots, & N=6\\ 1+192q+9216q^2+213248q^3+\cdots, & N=8\end{cases} \tag{4.56}$$

$$k=3: \quad \sigma(\mathcal{L}M) = \begin{cases} -5 - 160q - 1280q^2 - 7040q^3 + \cdots., & N=4 \\ 19 - 224q - 5376q^2 - 67200q^3 + \cdots, & N=6 \\ -53 - 1984q - 29696q^2 - 455936q^3 + \cdots, & N=8 \end{cases} \tag{4.57}$$

$$k=5: \quad \sigma(\mathcal{L}M) = \begin{cases} -35 - 1120q - 8960q^2 - 49280q^3 + \cdots, & N=4 \\ 341 + 2016q + 48384q^2 + 604800q^3 \cdots, & N=6 \\ -3219 - 101952q - 764928q^2 - 3134208q^3 + \cdots, & N=8 \end{cases} \tag{4.58}$$

(4.56),(4.57),(4.58) differ completely from those of the LG theory (3.38),(3.42),(3.43).

5 $\hat{A}$ genus

Now let us turn to the discussion of the $\hat{A}$ genus. $\hat{A}$ is defined for real $4n$ dimensional spin manifolds and hence we consider hypersurfaces with even N and k. In the LG formulation the elliptic genus for $\hat{A}$ is defined by

$$\chi_{\hat{A}}(\tau) = \frac{-i(-1)^{\frac{N}{2}}}{k} \sum_{a,b=0}^{k-1} (-1)^{a+b} \left(\frac{\theta_3(\frac{\tau+1}{2k} + \frac{b}{k} + \frac{a\tau}{k} \mid \tau\ )}{\theta_1(\frac{\tau+1}{2k} + \frac{b}{k} + \frac{a\tau}{k} \mid \tau\ )} \right)^N . \tag{5.59}$$

Basically θ_2 in χ_σ is replaced by θ_3 in $\chi_{\hat{A}}$. The sign $(-1)^{a+b}$ is fixed again by the consideration of modular invariance.

On the other hand the geometrical definition of the $\hat{A}$ elliptic genus is given by

$$\hat{A}(\mathcal{L}M) = q^{-\frac{d}{8}} \int_M ch(\tilde{\mathbf{E}}_q)\hat{A}(M) \tag{5.60}$$

where $\tilde{\mathbf{E}}_q$ is the bundle

$$\tilde{\mathbf{E}}_q = (\otimes_{n\geq 1}(\Lambda_{q^{n-\frac{1}{2}}} TM)) \otimes (\otimes_{n\geq 1}(\Lambda_{q^{n-\frac{1}{2}}} T^*M)) \otimes (\otimes_{n\geq 1} S_{q^n}(TM \oplus T^*M)). \tag{5.61}$$

We now have a half-integer moding for fermionic contributions corresponding to the NS boundary condition. $\hat{A}(M)$ is the classical $\hat{A}$ genus given by

$$\hat{A}(M) = \prod_{i=1}^{d} \frac{\frac{u_i}{2}}{\sinh \frac{u_i}{2}}. \tag{5.62}$$

where d is the complex dimension of the manifold. $\hat{A}$ polynomial is expanded as

$$\frac{\frac{u}{2}}{\sinh\frac{u}{2}} = \exp\Big(\sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{2j\cdot(2j)!}B_j u^{2j}\Big). \tag{5.63}$$

We next introduce an auxiliary function

$$h(q,z) = \prod_{n=1}\frac{(1+e^{2\pi iz}q^{n-\frac{1}{2}})(1+e^{-2\pi iz}q^{n-\frac{1}{2}})}{(1-e^{2\pi iz}q^{n})(1-e^{-2\pi iz}q^{n})} \tag{5.64}$$

and define

$$H_{2j}(q) = \frac{1}{2}(\frac{1}{2\pi i}\frac{d}{dz})^{2j}\log h(q,z)\,|_{z=0} \quad j=1,2,3\cdots. \tag{5.65}$$

We then have an identity

$$\exp(\frac{u}{2\pi i}\frac{d}{dz})h(q,z)|_{z=0} = h(q)\exp\left(\sum_{j=1}^{\infty}\frac{2}{(2j)!}H_{2j}(q)u^{2j}\right) \tag{5.66}$$

where

$$h(q) = h(q,z)|_{z=0}. \tag{5.67}$$

Then the elliptic genus is represented as

$$\hat{A}(\mathcal{L}M) = q^{-\frac{d}{8}}\int_M h(q)^d\exp\left(\sum_{j=1}^{\infty}\frac{2}{(2j)!}\Big((-1)^{j-1}\frac{B_j}{4j}+H_{2j}(q)\Big)\Big(\sum_{i=1}^{d}u_i^{2j}\Big)\right). \tag{5.68}$$

We have evaluated $\hat{A}$ genus for LG theory and compared the results from geometry. We have checked the exact agreements between them for $k=2,4,6$ and various even values of N.

6 discussions

In this article we have discussed a possible generalization of Calabi-Yau/Landau-Ginzburg correspondence to a more general class of manifolds. We have considered the case of degree-k hypersurfaces M_N^k in the complex projective space $\mathbf{CP}^{N-1}$. M_N^k has a positive, zero and negative 1st Chern class depending on $N>k$,$N=k$ and $N<k$, respectively. In all these cases we have found that the CFT/LG system always predicts the correct topological invariants when k =even and M_N^k's are spin manifolds. This fulfills our expectation that smooth quantum deformation or renormalization flow preserves the topological characteristics of the manifolds. On the other hand, when k =odd and manifold does not have spin structure, LG predictions for the signature are in disagreement with the geometry. Thus we have found some evidence that the CFT/geometry correspondence may exist in the case of spin manifolds.

In order to test this conjecture we have considered the following example of hypersurfaces in weighted projective space

$$\tilde{M}^k : X_1^{2k} + X_2^{2k} + X_3^k + X_4^k = 0. \tag{6.69}$$

These surfaces are discussed in the literature [16, 17]. First Chern class of $\tilde{M}^k$ is given by $3-k$ and $\tilde{M}^3$ is a K_3 surface. We have studied topological invariants of $\tilde{M}^k$ in the case of $k = 1, 3, 5$ for which the hypersurfaces have a spin structure and $k = 2, 4$ which are not spin manifolds.

We have computed the elliptic genus for the signature and have checked that in the case of $k = 1, 3, 5$ the LG prediction for the lower order q-expansion coefficients coincide exactly those of geometry. On the other hand, LG theory disagree with geometry in the case $k = 2, 4$. This example gives some additional evidence for our conjecture.

After finishing our computations we have found a brief remark in [3] (section 3) on the relevance of spin manifolds: there it is pointed out that in the case of non-vanishing first Chern class $U(1)_R$-symmetry of the theory becomes anomalous and the amplitude (3.10) is no longer a topological invariant. However, there is a remaining discrete $U(1)_R$-symmetry Z_{c_1} and (3.10) is still well-defined for $\gamma = \text{integer}/c_1$. Thus if c_1 is even and the manifold has a spin structure, the amplitude is well-defined at $\gamma = 1/2$ which corresponds to the Hirzebruch signature.

Our results are quite consistent with this remark: the LG computation works for any value of k in the case of Euler number ($\gamma = 0$), however, it works only for k =even in the case of signature.

It will be interesting to study the geometry/LG correspondence in more general examples of manifolds than hypersurfaces.

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Appendix A: Evaluation of the elliptic genus

In this appendix, we assume that both N and k are even integers. We first recall the definition of the auxiliary function $f(q,z)$

$$\begin{aligned} f(q,z) &= \prod_{j=1}^{\infty} \frac{(1+e^{2\pi i z}q^j)(1+e^{-2\pi i z}q^j)}{(1-e^{2\pi i z}q^j)(1-e^{-2\pi i z}q^j)} = \tan(\pi z)\frac{\theta_2(z|\tau)}{\theta_1(z|\tau)}, \\ f(q,0) &= \pi\frac{\theta_2(0|\tau)}{\theta_1'(0|\tau)}. \end{aligned} \tag{6.70}$$

By using

$$\theta_1'(0|\tau) = \pi\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau). \tag{6.71}$$

we find $f(q,0) = 1/(\theta_3(0|\tau)\theta_4(0|\tau))$. We start from the geometrical formula for the elliptic genus given in section 4,

$$\begin{aligned}&\sigma(\mathcal{L}M_N^k)\\ &= (2f(q,0))^{N-2}\int_{M_N^k}\exp\left(\sum_{j=1}^{\infty}\frac{2}{(2j)!}\left((-1)^{j-1}(2^{2j-1}-1)\frac{B_j}{2j}+F_{2j}(q)\right)(N-k^{2j})H^{2j}\right),\end{aligned} \tag{6.72}$$

$$F_{2j}(q) = \frac{1}{2}(\frac{1}{2\pi i})^{2j}\frac{d^{2j}}{dz^{2j}}\log(f(q,z))|_{z=0}. \tag{6.73}$$

Since $F_{2j+1}(q) = 0$ $(j \geq 0)$, we may sum the series in the exponent

$$\sum_{j=1}^{\infty}\frac{1}{(2j)!}F_{2j}(q)H^{2j} = \frac{1}{2}\left(\log f(q,\frac{H}{2\pi i}) - \log f(q,0)\right), \tag{6.74}$$

$$\sum_{j=1}^{\infty}\frac{1}{(2j)!}F_{2j}(q)k^{2j}H^{2j} = \frac{1}{2}\left(\log f(q,\frac{kH}{2\pi i}) - \log f(q,0)\right). \tag{6.75}$$

Sum over the Bernoulli numbers reproduces the Hirzebruch polynomial

$$\sum_{j=1}^{\infty}\frac{1}{(2j)!}(-1)^{j-1}(2^{2j-1}-1)\frac{B_j}{j}H^{2j} = \log\left(\frac{\frac{H}{2}}{\tanh\frac{H}{2}}\right), \tag{6.76}$$

$$\sum_{j=1}^{\infty}\frac{1}{(2j)!}(-1)^{j-1}(2^{2j-1}-1)\frac{B_j}{j}k^{2j}H^{2j} = \log\left(\frac{\frac{kH}{2}}{\tanh\frac{kH}{2}}\right). \tag{6.77}$$

We can then rewrite $\sigma(\mathcal{L}M_N^k)$ as

$$\begin{aligned}\sigma(\mathcal{L}M_N^k) &= (2f(q,0))^{N-2}\int_{M_N^k}\Big(\frac{\frac{H}{2}}{\tanh\frac{H}{2}}\Big)^N\Big(\frac{\tanh\frac{kH}{2}}{\frac{kH}{2}}\Big)\Big(\frac{f(q,\frac{H}{2\pi i})}{f(q,0)}\Big)^N\frac{f(q,0)}{f(q,\frac{kH}{2\pi i})}\\ &= \frac{(-i)^{N-1}}{2kf(q,0)}\int_{M_N^k}H^{N-1}\Big(\frac{\theta_2(\frac{H}{2\pi i}|\tau)}{\theta_1(\frac{H}{2\pi i}|\tau)}\Big)^N\frac{\theta_1(\frac{kH}{2\pi i}|\tau)}{\theta_2(\frac{kH}{2\pi i}|\tau)}.\end{aligned} \tag{6.78}$$

Since the integral $\int_{M_N^k}$ picks up the coefficient of the H^{N-2} term in the product of θ functions and $\int_{M_N^k}H^{N-2} = k$, we can express $\sigma(\mathcal{L}M_N^k)$ as

Lemma 1

$$\begin{aligned}\sigma(\mathcal{L}M_N^k) &= \frac{(-i)^{N-1}}{2}\theta_3(0|\tau)\theta_4(0|\tau)\oint_{C_{z=0}}\frac{dz}{2\pi i}\Big(\frac{\theta_2(\frac{z}{2\pi i}|\tau)}{\theta_1(\frac{z}{2\pi i}|\tau)}\Big)^N\frac{\theta_1(\frac{kz}{2\pi i}|\tau)}{\theta_2(\frac{kz}{2\pi i}|\tau)},\\ &= \frac{(-i)^{N-1}}{2}\theta_3(0|\tau)\theta_4(0|\tau)\oint_{C_{z=0}}dz\Big(\frac{\theta_2(z|\tau)}{\theta_1(z|\tau)}\Big)^N\frac{\theta_1(kz|\tau)}{\theta_2(kz|\tau)}.\end{aligned} \tag{6.79}$$

Here the integration contour $\oint_{C_{z=0}}$ circles around the origin. If we use the relation between the theta functions and the Weierstrass $\mathcal{P}$ function,

$$\Big(\frac{\theta_2(z|\tau)}{\theta_1(z|\tau)}\Big)^2 = \Big(\frac{\theta_2(0|\tau)}{\theta_1'(0|\tau)}\Big)^2(\mathcal{P}(z|\tau)-e_1(\tau)), \tag{6.80}$$

we can rewrite (6.79) as

$$\sigma(\mathcal{L}M_N^k) = \frac{(-i)^{N-1}}{2\pi^{N-1}}\frac{1}{(\theta_3(0|\tau)\theta_4(0|\tau))^{N-2}}\oint_{C_{z=0}} dz\frac{(\mathcal{P}(z|\tau)-e_1(\tau))^{\frac{N}{2}}}{(\mathcal{P}(kz|\tau)-e_1(\tau))^{\frac{1}{2}}}. \tag{6.81}$$

e_1 denotes one of the zeros of the cubic equation representing the elliptic curve $y^2 = 4x^3-g_2x-g_3 = 4(x-e_1)(x-e_2)(x-e_3)$. Then the task of evaluating $\sigma(\mathcal{L}M_N^k)$ is reduced to finding zeros of the function $\mathcal{P}(kz|\tau)-e_1(\tau)$ in the z-plane.

Let us introduce the function $X(z)$

$$X(z) \equiv \mathcal{P}(z) - e_1, \tag{6.82}$$
$$\frac{d}{dz}X(z) = \frac{d}{dz}\mathcal{P}(z). \tag{6.83}$$

(τ dependence is suppressed). We recall the addition theorem of the $\mathcal{P}$ function

$$\mathcal{P}(2z) = \frac{\Big(6\mathcal{P}(z)^2-\frac{g_2^2}{2}\Big)^2}{4\Big(4\mathcal{P}(z)^3-g_2\mathcal{P}(z)-g_3\Big)} - 2\mathcal{P}(z). \tag{6.84}$$

The coefficient functions g_2,g_3 of the elliptic curve are related to θ–constants as

$$g_2 = \frac{2}{3}\pi^4(u^8+v^8+w^8), \quad g_3 = \frac{4}{27}\pi^6(v^4-w^4)(2u^8+v^4w^4) \tag{6.85}$$

where u,v,w are defined by

$$u = \theta_3(0|\tau), \;\; v = \theta_4(0|\tau), \;\; w = \theta_2(0|\tau). \tag{6.86}$$

We can evaluate the elliptic genus in the case $k=2$ by using the above addition formula. If we want to compute the genus for $k=4$, we have to use the addition formula once more and express $\mathcal{P}(4z)$ in terms of $\mathcal{P}(z)$. If we represent the resulting expression in terms of $X(z)$, we have

Lemma 2

$$X(4z) = 16\pi^4\frac{(uv)^4}{(\frac{d}{dz}X(z))^2}\cdot\frac{\Big(R_4(X(z))\cdot S_4(X(z))\Big)^2}{\Big(R_4'(X(z))\cdot S_4(X(z)) - R_4(X(z))\cdot S_4'(X(z))\Big)^2}. \tag{6.87}$$

Here R, S are degree-4 polynomials of X

$$R_4(X) = X^4 - 4\pi^2 u^2 v^2 X^3 - \pi^4(4u^6v^2 + 4u^2v^6 + 2u^4v^4)X^2 - 4\pi^6 u^6 v^6 X + \pi^8 u^8 v^8, \tag{6.88}$$

$$S_4(X) = X^4 + 4\pi^2 u^2 v^2 X^3 + \pi^4(4u^6v^2 + 4u^2v^6 - 2u^4v^4)X^2 + 4\pi^6 u^6 v^6 X + \pi^8 u^8 v^8, \tag{6.89}$$

and $'$ means the derivative in X. We then have

$$\sigma(\mathcal{L}M_N^4) = \frac{(-i)^{N-1}}{8\pi^{N+1}(uv)^N} \oint_C dX \cdot X^{\frac{N}{2}} \cdot \frac{R_4'(X)S_4(X) - R_4(X)S_4'(X)}{R_4(X)S_4(X)} \tag{6.90}$$

$$= \frac{(-i)^{N-1}}{8\pi^{N+1}(uv)^N} \oint_C dX \cdot X^{\frac{N}{2}} \cdot \sum_{i=1}^{4} \left(\frac{1}{X - \alpha_i} - \frac{1}{X - \beta_i}\right) \tag{6.91}$$

$$= -\frac{(-i)^N}{4\pi^N (uv)^N} \left(\sum_{i=1}^{4} \alpha_i^{\frac{N}{2}} - \sum_{i=1}^{4} \beta_i^{\frac{N}{2}} \right). \tag{6.92}$$

Here α_i's (resp. $\beta_i's$) denote the roots of the algebraic equation $R_4(X) = 0$ (resp. $S_4(X) = 0$).

Note that the numerator of the right-hand-side of the addition formula (6.87) is a polynomial $T_{16}(X)$ of order 16 which is factored into a product of squares of $R_4(X)$ and $S_4(X)$. Thus each of the roots α_i and β_i have a multiplicity 2 in $T_{16}(X)$. On the other hand, if one substitutes $z = 1/8 + (b + a\tau)/4,\ a, b \in \mathbf{Z}$ into (6.87), the left-hand-side vanishes (recall the relation $\mathcal{P}(1/2 + b + a\tau|\tau) = e_1(\tau)$). Thus we recognize that $\{X = X(1/8 + (b + a\tau)/4|\tau),\ a, b = 0, 1, 2, 3\}$ are the solutions of the algebraic equation $T_{16}(X) = 0$. From the definition (6.82) $X(1/8 + (b + a\tau)/4|\tau) = X(1/8 + (4 - b - 1 + (4 - a)\tau)/4|\tau)$ and hence each of the roots has a multiplicity 2. Then by reducing the range of b to $b = 0, 1$, the roots $\{X = X(1/8 + (b + a\tau)/4|\tau),\ a = 0, 1, 2, 3,\ b = 0, 1\}$ are in one to one correspondence with $\{\alpha_i,\ \beta_i,\ i = 1, 2, 3, 4\}$.

Explicitly the zeros of $R_4(X) = 0$ and $S_4(X) = 0$ are given by

$$\begin{cases} \alpha_j = \pi^2 uv(uv - (u^2 + v^2) \mp (u - v)\sqrt{u^2 + v^2}), & j = 1, 2, \\ \alpha_j = \pi^2 uv(uv + (u^2 + v^2) \mp (u + v)\sqrt{u^2 + v^2}), & j = 3, 4. \end{cases} \tag{6.93}$$

$$\begin{cases} \beta_j = \pi^2 uv(- uv - i(u^2 - v^2) \pm (u - iv)\sqrt{v^2 - u^2}), & j = 1, 2, \\ \beta_j = \pi^2 uv(- uv + i(u^2 - v^2) \pm (u + iv)\sqrt{v^2 - u^2}), & j = 3, 4, \end{cases} \tag{6.94}$$

We find that $\{\alpha_i, i = 1, 2, 3, 4\} = \{X(1/8 + (b + a\tau)/4|\tau), (a = 0, 2, b = 0, 1)\}$, $\{\beta_i, i = 1, 2, 3, 4\} = \{X(1/8 + (b + a\tau)/4|\tau), (a = 1, 3, b = 0, 1)\}$.

Thus we finally prove

Theorem 1

$$\sigma(\mathcal{L}M_N^4)(\tau) = -\frac{(-1)^{\frac{N}{2}}}{4} \sum_{a=0}^{3} \sum_{b=0}^{3} (-1)^a \left(\frac{\theta_2(\frac{1}{8} + \frac{b + a\tau}{4}|\tau)}{\theta_1(\frac{1}{8} + \frac{b + a\tau}{4}|\tau)} \right)^N \tag{6.95}$$

This establishes the equivalence of LG theory with geometry for the case $k = 4$.

By substituting (6.93),(6.94) into (6.92) we obtain the results (3.40) of section 3.

Remarks

The $k = 2$ case is easy and we leave this case to the reader. We can prove the $k = 6$ version of Theorem 1 along the same way, using the explicit calculation of the addition formula for $X(6z|\tau)$. The case of $\hat{A}$-genus can also be proved using the same method. In general the proof of the equivalence of LG theory with geometry is reduced to the proof of the analogue of Lemma 2 for general even values of k.

In the case of odd k we lose the factorization of the polynomial in the numerator of the right-hand-side of the addition theorem. This necessarily happens since the polynomial is degree k^2 which is not divisible by 2. Then we can not express $\sigma(\mathcal{L}M_N^k)$ as a residue integral in the X variable.

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